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A note on symmetric connections

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Abstract

In this paper we analyze a reciprocal of the fundamental theorem of Riemannian geometry. We give a condition for a symmetric connection to be locally the Levi-Civita connection of a metric. We also construct a couple of natural examples of connections on the *n*-dimensional torus and investigate the global problem.

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1. Introduction

The problem of the determination of a metric compatible with a given symmetric connection ∇ in the tangent bundle TM of a smooth manifold M has some recent history. The problem was investigated mainly by mathematical physicists like Thompson in [5] and [6] and Edgar in [2]) and [3]. In his paper [2] Edgar gives necessary and sufficient conditions for a volume preserving connection (which in our invariant formulation is equivalent with

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the vanishing of the "Chern" form associated to the connection) to be locally the Levi-Civita connection of a metric. In this paper we give an example of a connection on the n-dimensional torus which satisfies the conditions of his theorem, but which is not globally metric. Global natural questions also arise and we address this question by analyzing a natural map. This map can be described briefly as the map which assigns a connection ∇ to a Riemannian metric g. We analyze the linearization of this map, describing precisely what the kernel is. The finite dimensionality of the kernel of this map is established. As Edgar notes there are two essential conditions for a symmetric connection to be locally metric. One is that the trace of the curvature form has to be zero, and the other one is the existence of a Bianchi tensor. These are, according to Edgar [2], sufficient and necessary conditions for the connection to be locally metric. In order to investigate the global existence of a metric one has to take into account the holonomy group of the connection. The earliest result in this direction was obtained by Schmidt in [4] where he proved the following:

Theorem 1.1 (Schmidt [4]). *Let ∇ be a symmetric connection in* TM. *If the holonomy group of the connection is $\mathcal{H}$ and is a subgroup of the orthogonal group* $\mathrm{O}(n)$, *then ∇ is the Levi-Civita connection of a Riemannian metric.*

Let us investigate the local problem first. Let ∇ denote a symmetric connection on the tangent bundle of a smooth manifold M^n, $(e_i)_{i=1}^n$ a local frame and (θ_i^j) the connection forms of ∇ (i.e. $\nabla e_i = \theta_i^j e_j$). As a consequence of Cartan's structural equation

$$\mathrm{d}\theta_i^j = \Omega_i^j - \theta_i^k \wedge \theta_k^j,$$

we see that the trace of the curvature form

$$\mathrm{Tr}(\Omega) = \sum_{i=1}^{n} \Omega_i^i$$

is independent of the choice of the frame and we shall call this two-form the Chern form associated to the connection. The reason is that the cohomology class of this form is independent of the connection and it is also called the first Chern class of the manifold. If the connection is locally a metric connection then the Cartan equation can be rewritten as

$$\Omega_{ij} = \mathrm{d}\theta_{ij} - \theta_{ik} \wedge \theta_{kj},$$

and the curvature matrix in this case being skew symmetric it follows that the trace (which is invariant) has to be zero. Thus, we obtain a necessary condition for a symmetric connection to be locally metrizable. We shall see that this is not sufficient for the local metrizability of a connection. A very restrictive sufficient condition is given by the following lemma:

Lemma 1.1. *Let M be a smooth manifold and ∇ a symmetric flat connection. Then ∇ is locally metrizable.*

The proof of the lemma is elementary and therefore we omit it.

2. Basic examples

The following is an example of a connection with vanishing Chern form which is not even locally a metric connection. Let $E = \partial\theta$ and $F = \partial\psi$ be the two vector fields on $T^2 = S^1 \times S^1$ defined by the two angles θ and ψ corresponding to the two circles and define ∇ as

$$\nabla_E E = \nabla_F F = 0, \qquad \nabla_E F = \nabla_F E = F. \tag{1}$$

It is easy to see that if this were globally a metric connection the sectional curvature should be positive, thus, in contradiction with the Euler characteristic of T^2. In what follows we will see that ∇ defined as above cannot be locally the Levi-Civita connection of a metric.

Proposition 2.1. *Let ∇ defined as in(1). Then the Chern form of the connection is zero but ∇ is not locally a metric connection.*

Proof. First we compute the Chern form of ∇. The connection forms with respect to the frame $E = \partial\theta$, $F = \partial\psi$ are defined by the relations

$$\nabla E = \theta_1^1 E + \theta_1^2 F,$$

and

$$\nabla F = \theta_2^1 E + \theta_2^2 F.$$

Hence

$$\theta_1^1 = 0, \qquad \theta_1^2 = \mathrm{d}\psi, \qquad \theta_2^1 = 0, \qquad \theta_2^2 = \mathrm{d}\theta$$

so it follows that the curvature forms

$$\Omega_1^1 = \Omega_2^2 = 0.$$

Thus the Chern form is zero. Next we show that this connection cannot be locally a metric connection. The problem is equivalent to proving the same thing for the connection in $\mathbb{R}^2$. By choosing an appropriate chart we can actually "identify" $\partial\theta$ with ∂x and $\partial\psi$ with ∂y. If this connection were locally a metric connection we could actually recover the metric by parallel transporting the vectors ∂x and ∂y along radii emanating from the origin and then defining these vector fields as being orthonormal. After doing, this one computes the coefficients of this metric with respect to the (x,y) coordinates and obtains

$$g_{11} = \frac{2s^2 - 2s + 1}{(1-s)^2}, \qquad g_{22} = \frac{1}{(1-s)^2}, \qquad g_{12} = g_{21} = \frac{s}{(1-s)^2}.$$

Here $s = x + y$ and the coefficients of the inverse metric are

$$g^{11} = 1, \qquad g^{22} = 2s^2 - 2s + 1, \qquad g^{12} = g^{21} = -s.$$

From the relations which define the connection it is obvious that $\Gamma^1_{11} = 0$, while the corresponding Christofell symbol of the metric (g_{ij}) is different from zero in a neighborhood of the origin. □

An example of a flat connection (hence locally metric) can be constructed on the n-dimensional torus as follows. Let

$$T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n\text{-}times},$$

and $\theta_1, \theta_2, \ldots, \theta_n$ be the corresponding angles on the circles. Then, we have the following global vector fields $E_i = \partial\theta_i$ for $i = 1, n$. We define the connection through its action on these vector fields, namely:

$$\nabla_{E_i} E_i = E_i, \quad \text{for} \quad i = 1, n \quad \text{and} \quad \nabla_{E_i} E_j = \nabla_{E_j} E_i = 0 \quad \text{for} \quad i \neq j. \tag{2}$$

The following proposition shows that this is a locally metrizable connection which is not globally metrizable.

Proposition 2.2. *Let ∇ defined as in (2) on T^n. Then ∇ is a flat connection which is not globally a metric connection.*

Proof. First we shall see that the connection is flat. Let $R(X,Y)Z$ denote the full Riemannian tensor. Then

$$R(E_i, E_j)E_i = \nabla_{E_i}\nabla_{E_j}E_i - \nabla_{E_j}\nabla_{E_i}E_i = 0 - 0 = 0,$$

hence $R \equiv 0$. Here we have used the fact that E_i and E_j commute.

Next we shall prove that ∇ is not globally the connection associated to a metric. Let $E_1 = \partial\theta_1$ and let us assume the contrary. Assume $g(X, Y) = \langle X, Y\rangle$ is a Riemannian metric having ∇ as its Levi-Civita connection. Then

$$E_1\langle E_1, E_1\rangle = \langle 2\nabla_{E_1}E_1, E_1\rangle = 2\langle E_1, E_1\rangle.$$

Now let $\gamma = \gamma(t)$ be a flow line of the vector field E_1 (this flow line is actually just one of the circles which generates the torus) and let $f(t) = |E_1(\gamma(t))|^2$. Then obviously $-\infty < t < +\infty$ and f satisfies the differential equation

$$f' = 2f,$$

which, in turn, implies that f is not bounded (being an exponential). Hence, ∇ cannot be globally a metric connection although, since is flat, it is locally metrizable according to Lemma 1.1. □

Next section addresses the global problem by investigating a natural map.

3. The Levi-Civita map

Globally we ask the following natural questions:

1. Are there any manifolds where any locally metric connection is globally metrizable?
2. Under what topological assumptions is the map

$$g \to \nabla_g$$

surjective onto the space of all locally metrizable connections?

Here, g denotes a Riemannian metric on M and ∇_g its associated Levi-Civita connection. Before stating the main global result of the paper we need to introduce more structure. In what follows M^n is a compact manifold with a reference Riemannian metric λ and with associated Levi-Civita connection D. Let $\mathcal{T}$ denote the completion of the space of all smooth positive definite symmetric tensor fields of type (2, 0) endowed with the norm

$$\|h\|_1 = \sup_{p\in M} |h(p)| + \sup_{p\in M} |(Dh)(p)|. \tag{3}$$

Let

$$\mathbb{B} = \{\nabla - D | \nabla \text{ is a symmetric connection on TM}\},$$

endowed with the norm defined as

$$\|B\| = \sup_{p\in M} |B(p)|. \tag{4}$$

Here, the pointwise length of tensors is taken with respect to the reference metric λ. The choice of the metric λ is not important since any other metric will produce equivalent norms. It is obvious that $\mathbb{B}$ is a subset of the space of smooth symmetric tensor fields of type (2, 1). Let $\mathcal{B}$ be the completion of $\mathbb{B}$. Having defined all of the above, we observe that the Levi-Civita map is a mapping between the open set $\mathcal{O}$ of positively definite tensors in $\mathcal{T}$ and $\mathcal{B}$. This mapping can be interpreted as a parametrization of the space of metric connections. A more detailed topological analysis of the Levi-Civita map is done by Williams in [7]. The following global result refers to the properties of the mapping:

$$\mathcal{L} : \mathcal{O} \to \mathcal{B},$$

where $\mathcal{L}(g) = \nabla_g - D$ and ∇_g is the Levi-Civita connection of g.

Theorem 3.1. *Let M^n be a compact smooth manifold. Then the map $\mathcal{L}$ defined as above is of class C^1 and its differential has finite dimensional kernel. Moreover, the kernel of the differential of the map at the point $g \in \mathcal{T}$ is the space of all the symmetric tensors of type* (2, 0) *which are parallel with respect to the Levi-Civita connection of g.*

Proof. From the definition of $\mathcal{O}$ and $\mathcal{L}$ it is easy to see that the map is of class C^1. Now let us denote by $T_g\mathcal{L}$ the differential of $\mathcal{L}$ at the point $g \in \mathcal{O}$. If h is an arbitrary vector in $\mathcal{T}$ then

$$T_g\mathcal{L}(h) = \lim_{t\to 0} \frac{1}{t}(\mathcal{L}(g+th) - \mathcal{L}(g)). \tag{5}$$

First, we note that for small values of the parameter t the symmetric tensor $g + th$ is a Riemannian metric and consequently, we have a unique connection associated to it, namely ∇_{g+th}. From the definition of the mapping $\mathcal{L}$ we have

$$\mathcal{L}(g+th) - \mathcal{L}(g) = \nabla_{g+th} - \nabla_g. \tag{6}$$

Taking into account (5) and (6), and the fact that $\mathcal{L}$ is of class C^1, and since all of the operations involved are continuous we note that

$$g(T_g\mathcal{L}(h)(X,Y),Z) = \lim_{t\to 0} \frac{1}{t}(g+th)((\nabla_{g+th} - \nabla_g)(X,Y),Z), \tag{7}$$

for all vector fields $X, Y, Z \in$ TM. Let us now concentrate on the right-hand side of (7). We have

$$\begin{aligned}(g+th)((\nabla_{g+th} - \nabla_g)(X,Y),Z) = {}& (g+th)((\nabla_{g+th}(X,Y),Z) - g(\nabla_g(X,Y),Z)\\ &-th(\nabla_g(X,Y),Z),\end{aligned}$$

and as in [1, p. 234, Eq. (9)]we have the following:

$$\begin{aligned}(g+th)((\nabla_{g+th}(X,Y),Z) = {}& \tfrac{1}{2}\{X((g+th)(Y,Z)) + Y((g+th)(Z,X))\\ &-Z((g+th)(X,Y)) - (g+th)([X,Y],Z)\\ &-(g+th)([Y,Z],X) + (g+th)([Z,X],Y)\},\end{aligned} \tag{8}$$

as well as

$$\begin{aligned}g((\nabla_g(X,Y),Z) = {}& \tfrac{1}{2}\{Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g([X,Y],Z)\\ &-g([Y,Z],X) + g([Z,X],Y)\}.\end{aligned} \tag{9}$$

It follows that the right-hand side of (7) becomes

$$\begin{aligned}&\lim_{t\to 0} \frac{1}{t}(g+th)((\nabla_{g+th} - \nabla_g)(X,Y),Z)\\ &\quad = \frac{1}{2}\{X(h(Y,Z)) + Yh(Z,X) - Zh(X,Y) - h([X,Y],Z) - h([Y,Z],X)\\ &\qquad +h([Z,X],Y)\} - h(\nabla_g(X,Y),Z),\end{aligned} \tag{10}$$

hence

$$\begin{aligned}g(T_g\mathcal{L}(h)(X,Y),Z) = {}& \tfrac{1}{2}\{X(h(Y,Z)) + Yh(Z,X) - Zh(X,Y) - h([X,Y],Z)\\ &-h([Y,Z],X) + h([Z,X],Y)\} - h(\nabla_g(X,Y),Z).\end{aligned} \tag{11}$$

We note that (11) implicitly defines $T_g\mathcal{L}(h)$. All we need to prove now is that if $h \in \ker(T_g\mathcal{L})$, then actually h is parallel with respect to ∇_g. Let $h \in \ker(T_g\mathcal{L})$, then according to (11)

$$\tfrac{1}{2}\{X(h(Y,Z)) + Yh(Z,X) - Zh(X,Y) - h([X,Y],Z) - h([Y,Z],X) + h([Z,X],Y)\} = h(\nabla_g(X,Y),Z), \tag{12}$$

for every $X, Y, Z \in$ TM. Now let us compute $\nabla_g h$. We have

$$\nabla_g(X,h)(Y,Z) = X(h(Y,Z)) - h(\nabla_g(X,Y),Z) - h(Y,\nabla_g(X,Z)). \tag{13}$$

Here X,Y,Z are arbitrary tangent vector fields and $\nabla_g(X,h)(Y,Z)$ is the covariant derivative of the h tensor in the direction X. Applying (12) once again and permuting Z and Y we have

$$\tfrac{1}{2}\{X(h(Y,Z)) + Zh(Y,X) - Yh(X,Z) - h([X,Y],Z) - h([Z,Y],X) + h([Y,X],Z)\} = h(\nabla_g(X,Z),Y). \tag{14}$$

Now taking into account (12) and (14) we observe that the right-hand side of (13) is zero, hence

$$\nabla_g h = 0.$$

This proves the theorem. □

Finally, we point out that in the event that the cokernel of the Levi-Civita map is finite dimensional the index of the map is a topological invariant. Heuristically, the dimension of the kernel of this map quantifies the metrics compatible with a given metric connection, whereas the dimension of the cokernel of this map measures how far from being globally metric is a locally metrizable connection . Thus, the latter is again a topological invariant. In addition, it follows that the dimension of the kernel of the Levi-Civita map has to be itself topological. Notably, relating the dimension of the kernel or the index of the Levi-Civita map with the usual topological invariants of the manifold is an interesting question in itself.

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